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AP Calculus

13 April 2015

Sequences and Series

Sequences are introduced to students early on, before they know that what they are reciting is a “sequence”. This begins, of course, when we learn to count. Sequences get more complex as we study them, and they have patterns that may be hard to see when looking at one term by term. Sequences have many important uses and applications that become apparent as you see how many different things they can actually do.

A *sequence* is an ordered set of numbers. These numbers can be organized in many different ways. They could have a common difference, where adding or subtracting a certain amount would get you to the next term, and all terms following. This is called an *arithmetic* sequence. They could also have a common ratio, where multiplying or dividing by a certain amount would get you to the next term, and all terms following. This is called a *geometric* sequence. Sequences can be finite, where they eventually stop at a given point; however, all of these sequences can be extended infinitely to go on forever.

A *series* is the sum of all of the terms in a sequence. So, sequences and series do have something huge in common: they will always have the same terms. Whether you are working with an arithmetic or geometric sequence, the terms will remain the same in a series after having been worked out into a sequence. However, the two have that huge difference as well: a series can technically be boiled down to a single number, finite or infinite. To *sum it all up*, a series takes all of the terms in a sequence, and adds them all together.

The sequence of which a series is made can be finite or infinite, and the number that they all add up to could also be finite or infinite. If all of the numbers of an infinite sequence are added together to make one finite number, then the series is said to converge. On the other hand, if all of the numbers of a sequence are added together and become infinity, then the series is said to diverge. Now, a sequence can also converge or diverge. If the terms in a sequence get closer and closer to one number, then the sequence is said to converge. What really makes these properties interesting is that the terms in a converging sequence can be added up, and that same series will *diverge*. For example, consider the following sequence:

This sequence, when extended infinitely, converges to a finite number, two. However, when you add these terms together, they get closer and closer to infinity, because each subsequent term is larger than the previous term. The fifth partial sum, or the sum of the first five terms of the sequence, comes out to be 9.88889. If you were to keep adding terms, then this sum would get larger, and eventually approach infinity. This would mean that while the sequence *converges*, the same series will *diverge*.

When looking at transcendental functions, or functions that cannot be expressed with a finite sequence of algebraic operations, we often wonder where the values come from. There must be some formula that can be derived to calculate values from functions like or . The Taylor series approximation method can be used to make an *infinite* series of terms that when added together create the transcendental function. These series will always be in power form, where the terms are powers of x, translations of x, and other transformations of x. These Taylor series can be centered around any x-value, but when it is centered around , the Taylor series is called a *Maclaurin* series. This is the only difference between a Taylor and Maclaurin series. They are technically the same thing, but the Maclaurin series is just a special case of a Taylor series, much like a circle is a special case of an ellipse.

Deriving a Taylor or Maclaurin series is pretty straightforward. We can start with the Taylor series. We must make two assumptions; the first being that the function has a power series representation. So, let’s consider the standard form of a power series:

We should also assume that has derivatives of every order and that we can find all of them. Assuming these two things, all we then need are the coefficients, . This is actually fairly straightforward. All we need to do is evaluate everything at . This would give the following:

Now we know what is! Sadly, there is no value to plug into the function that will easily tell us what the other coefficients are. However, if we take the *derivative* of , we get the following:

Evaluating this at would give . Continuing with this idea, the second derivative would be the following:

Evaluating this at gives or . Using the third derivative would give the following:

Therefore, or . Using the fourth derivative would give the following:

Therefore, or . Now, we start to notice a pattern. In general, the coefficients can be given by the following:

This even works with because . With this information, we can conclude that the Taylor series for about is:

Knowing this, we also know the Maclaurin series for as it is the same thing, but centered at . This would give the following:

For example, the Maclaurin series for would be given by the following:

In contrast, the Taylor series for , centered about would be given by the following:

Luckily, every derivative of is itself, so the coefficient for each term would only be .

Again, the only difference between a Taylor series and a Maclaurin series is that a Maclaurin series is centered about and a Taylor series is centered about . Other than that, they both approximate transcendental functions with an infinite power series.

Taylor and Maclaurin series *approximate* functions. For example, if we take the Maclaurin series for and evaluate it at , we would get an approximation for the value of . Taking the eighth partial sum of the Maclaurin series, this is shown by the following:

The actual value of is 7.38905609893. These values are close, but not exactly the same. In other words, there is some error in the approximation. There are three different types of error – actually computing error, LaGrange error, and alternating series error. The *actual* error is easy to compute – just take the actual value and subtract the value obtained by the approximation.

Figure 1. Actual error calculation

Figure 1 shows the calculation of the actual error in the approximation of by the eighth partial sum of the Maclaurin series for . The approximation is off by 0.00810371798, which tells us that the approximation is good to the second decimal place. The LaGrange form of the remainder estimates the accuracy of an approximation by finding a maximum value of the (n+1)th derivative and evaluating the first term of the tail. In other words, if is the maximum value of on an interval between and , then the LaGrange error bound is given by .

Figure 2. LaGrange error calculation

Figure 2 shows the calculation for the LaGrange error of the approximation for . This value tells us that the approximation is accurate to no less than one decimal place, which we know to be true, because the actual error showed that the approximation was accurate to the second decimal place.

One last type of error remains, and that is the alternating series error. For this, you must have an *alternating* series, or a series wherein the terms alternate positive and negative. For example, the Taylor series for , centered about , which is given by the following:

Estimating using the third partial sum would give the value as it is in Figure 3 below.

Figure 3. Estimation of ln(1.4)

Figure 3 shows the estimation of using the third partial sum of the Taylor series for . The alternating series error is easy to find, as it is just the absolute value of the first term of the tail. Before we calculate this, we must check to see if the alternating series test applies. The first assumption, that the series does in fact alternate, is met. The second assumption, that , is met. Finally, the third assumption, that each subsequent term is less than the term before it in absolute value, is also met. The alternating series error can now be calculated.

Figure 4. Alternating series error calculation

Figure 4 shows the alternating series error calculation for from the Taylor series for . The value tells us that the approximation is good to at least two decimal places. We can check this by finding the actual error. The actual error would be . This value comes out to be -0.00486, which confirms the alternating series error that the approximation is accurate to at least two decimal places.

Now that we have gained an understanding of the topics, let’s apply these properties and rules to problems from an AP Calculus test!

**The Problems**

Problem 1 gives the following scenario: “The function *f* is defined by the power series for all real numbers x for which the series converges.”

Part (a) of Problem 1 states the following: “Find the interval of convergence of the power series for *f*. Justify your answer.” When asked to find the interval of convergence, they are implying that you must use the ratio test. Applying the ratio test to this problem would give the following:

Therefore, we can state the following:

So,

The interval of convergence could be , but we must check the endpoints to see if the series converges at either value. At , diverges by the nth term test, because . At , diverges because the series alternates between 1 and -1, never converging to one value. Therefore, the interval of convergence is .

Part (b) of Problem 1 states the following: “The power series above is the Taylor series for *f* about . Find the sum of the series for *f*.” This is actually pretty straightforward because we were given a geometric series, with a common ratio of (x + 1). That means that the sum of the series is , which comes from the fact that the sum of any geometric series is simply , where *a* is the coefficient and *r* is the common ratio.

Part (c) of Problem 1 states the following: “Let *g* be the function defined by . Find the value of , if it exists, or explain why cannot be determined.” This problem will be worked out in Figure 5 below.

Figure 5. Calculation for Problem 1, part (c)

Figure 5 shows the calculation for part (c), which shows that does in fact exist and is approximately 0.6931.

Part (d) of Problem 1 states the following: “Let be the function defined by . Find the first three nonzero terms and the general term of the Taylor series for *h* about . Find the value of .” All that is needed in this problem is the substitution of into . Doing this would give us the following series:

Now, we just have to plug in 0.5 into this series. This is given by the following:

The value of , when evaluated by using the series, is simply .

Problem 2 gives the following scenario: “Which of the following series diverge? Be sure to address which test you have chosen and WHY you chose it.”

Part (a) of Problem 2 gives the following series: . For this, we will use the limit comparison test, because no other test will prove its convergence or divergence; in other words, it is inconclusive in the other tests. We will compare this series to the harmonic series, which is known to diverge. If the limit of their ratio is infinity, then the series diverges. This problem will be worked out in Figure 6 below.

Figure 6. Work for Problem 2, part (a)

Figure 6 shows that the series given in part (a) of Problem 2 does in fact diverge because it was compared to a series known to diverge, and the limit of their ratios was infinity.

Part (b) of Problem 2 gives the following series: . For this, we will use the alternating series test, because by looking at the series, we see that it alternates. That is actually the first assumption, so we know that it is met. The second assumption is that , which is also met because the denominator would approach infinity, making the quotient approach zero. The third and final assumption is that each subsequent term is larger than the last, which is also met because the natural log function is always increasing, so the denominator will always increase, which means that the quotient will always decrease in value. Therefore, by the Alternating Series Test, this series converges.

Part (c) of Problem 2 gives the following series: . For this, we can use the geometric series test because it takes the form of , and it is obvious that the alternating series test will not work because , and the alternating series test does NOT prove divergence. The common ratio in this series is , which is greater than 1, so by the Geometric Series Test, this series diverges.

Problem 3 states the following: “Find the interval of convergence. Be sure to check the endpoints,” and gives the following series: . Again, as stated earlier, finding the interval of convergence implies using the ratio test. Applying the ratio test to this problem would give the following:

Now, the quotient can be dropped because, by L’hôpital’s Rule, it will approach 1 as *n* approaches infinity. We can then simplify what is left to get the following:

So,

The interval of convergence could be , but we must check the endpoints to be sure, as either endpoint could also allow the series to converge. When , you get the following series: . We can use the alternating series test to check and see if this converges. The first assumption, that the series alternates, is met. The second assumption, that , is also met. Finally, the third assumption, that each subsequent term is less than the previous, is also met. Therefore, the series converges when .

When , you get the following series: . Writing out the terms for this series would give you the following:

This just so happens to be the same as the harmonic series, which is known to diverge. Knowing this, we can conclude that the interval of convergence for the series for Problem 3 is .

Problem 4 states the following: “Write the 5th partial sum of the power series for . Use this partial sum to approximate . Discuss the error in your estimate.” This should be fairly straightforward, as the power series for is known. The fifth partial sum for that power series is given by the following:

To approximate , we simply plug in 0.1 into the fifth partial sum, as shown below:

We can conclude that is *roughly* 1.10517083333. Using the LaGrange form for error that we discussed earlier gives us the following:

This value tells us that the approximation is accurate to at least the seventh decimal place. We can check this by computing the actual error, as shown below:

The approximation is in fact accurate to the seventh decimal place, so the LaGrange form of the error is correct.